High-Intensity Quantum Electrodynamics. I. Intensity Dependences for Feynman Diagrams

L. M. FRANTZ
TRW Space Technology Laboratories, Redondo Beach, California
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The prescriptions for evaluating Feynman diagrams in quantum electrodynamics are extended to the regime of incident photon beams having nonzero intensity. This is done both for the time-dependent formulation of the S-matrix perturbation theory, and for the stationary-state Brillouin-Wigner perturbation development. The results are applicable to a theory of any boson field, as they depend only on the Bose-Einstein commutation properties of the field operators; no restrictions are made on the nature of the interaction Hamiltonian. It is shown that, as far as radiative corrections are concerned, the presence of the incident beam is to be ignored; it is to be manifested by a weighting factor which depends on the count of only external photon lines connected to the scatterer line. The distribution of the incident photons over modes of the radiation field is completely arbitrary, and the results are valid to all orders in the perturbation theory.

I. INTRODUCTION

THE quantum theory of intense photon beams has become of interest because of the advent of lasers.[1] The theory of the interaction of photons with charged particles has long been a subject of detailed investigation; quantum electrodynamics has been developed to a point of great sophistication and has been very successful in predicting essentially all hitherto measurable elementary phenomena in its realm. The existing formulations, however, have been systematized only in the limit of low intensities of incident photon beams. In the past this restriction has been no problem, since all available beams have been weak enough for the theory to be applicable. Recent developments in laser technology,[2] however, have made available photon-beam intensities which are so much greater than any previously dealt with that it now becomes necessary to re-examine the older theoretical formulations. It must be determined at what intensities they fail and what the high intensity phenomena actually are.

The present paper is devoted to the first phase of a program directed at these questions. Its objective is to display the explicit dependence of the perturbation term associated with any general Feynman diagram on the intensity of the incident photon beam. This will be done both for the covariant formulation of the S-matrix perturbation theory and for the stationary-state Brillouin-Wigner perturbation development. The emphasis will be on the latter. The reason is that one of the ultimate goals of this program is an understanding of resonance scattering from internally structured quantum entities like atoms, and it appears that the fact that the energy dependences are more explicitly displayed from the very beginning in the stationary-state development makes it more amenable to an investigation of such phenomena in the presence of intense beams. In a later paper, we shall discuss the properties of the modified propagator in this context with particular emphasis on its relevance to natural line shape and line shift and their intensity dependences. This paper will be confined to deriving the formal diagrammatic prescriptions for intensity dependences. No attempt will be made to calculate specific cross sections or to estimate magnitudes of correction terms; it is the author's belief that the basic formalism itself requires further development before reliable calculations can be made.

In Sec. II we derive the expressions for the intensity dependences to be associated with the general diagram in the time-dependent S-matrix formulation. The derivation is restricted to an interaction Hamiltonian which is linear in the boson field operator, as is the case in relativistic quantum electrodynamics. In all other respects the results are appropriate to a theory of any boson field, not just photons, since they depend only on the Bose-Einstein commutation properties of the creation and annihilation operators. We shall show that in general for each external photon line connecting to the charged particle line there is to be a factor roughly proportional to the square root of the photon density in the mode of the radiation field appropriate to the external photon line. This result is a generalization to arbitrary order in the perturbation theory of the well-known rule that a matrix element of a linear Hamiltonian for a transition from n to $n-1$ bosons in a given mode is proportional to $n^{1/2}$, and a matrix element for a transition from n to $n+1$ bosons is proportional to $(n+1)^{1/2}$. However, as far as conventional radiative corrections are concerned, the presence of the beam is to be ignored. *A priori*, this conclusion might not be expected at all. A superficial examination of a typical high intensity scattering calculation based on the above

[1] D. A. Kleiman, Phys. Rev. **125**, 87 (1962); **128**, 1761 (1962); R. Braunstein, *ibid.* **125**, 475 (1962); J. A. Armstrong, N. Bloembergen, J. Ducuing, and P. S. Pershan, *ibid.* **127**, 1918 (1962); Vachaspati, *ibid.* **128**, 664 (1962); R. L. Smith, *ibid.* **128**, 2225 (1962); P. Stehle, J. Opt. Soc. Am. **53**, 1003 (1963); Z. Fried, Phys. Letters **3**, 349 (1963); Z. Fried and J. H. Eberly, Bull. Am. Phys. Soc. **8**, 615 (1963); M. Mizushima, Phys. Rev. **132**, 951 (1963); L. S. Brown and T. W. B. Kibble, Phys. Rev. Letters **11**, A2 (1963); T. D. Holstein, Space Technology Laboratories, Report 9801-6008-KU-000 1963 (unpublished).

[2] F. R. Marshall, D. H. Roberts, and R. F. Wuerker, Bull. Am. Phys. Soc. **7**, 445 (1962).

rule can lead to an erroneous conclusion, viz., that radiative corrections to all orders are modified by the existence of an intense photon beam. For a portion of a diagram in which a virtual photon is simply emitted and reabsorbed, for example, one might expect the contribution to be of the form

$$\sum_k (n_k+1)f(k)/V,$$

where n_k is the number of photons before the emission in the mode labeled by k. The factor $1/V$ results from the fact that in periodic quantization each vertex operator is proportional to $1/V^{1/2}$. If, now, all the beam photons are in mode q, the density dependent term can be separated out, and the transition to the continuum accomplished by

$$\sum_k \rightarrow [V/(2\pi)^3]\int d^3k, \quad \text{as} \quad V\rightarrow\infty, \tag{1.1}$$

to give

$$(n_q/V)f(q)+(2\pi)^{-3}\int d^3k f(k).$$

If n_q remains finite as V approaches infinity, the extra term vanishes. However, if n_q also approaches infinity in such a way that the incident photon density $\rho=n_q/V$ remains finite, then there appears to be an additional term $(n_q/V)f(q)$ resulting from the presence of the beam. We shall see, though, that the sum of all such terms vanishes.

In Sec. III, we derive the expressions for the intensity dependences in the stationary-state S-matrix formulation. Here, for the sake of generality, we do not restrict ourselves to a linear interaction Hamiltonian. It can be of arbitrary order in the boson field operators: linear, bilinear, multilinear, or any superposition of orders. In addition, we consider the circumstances in which certain categories of Feynman diagrams are excluded by means of projection operators. The rules are found to be the same as for the time-dependent formulation in the sense that they continue to depend only on the count of external photon lines. Whereas in Sec. II all of the photons in the incident beam are taken to be confined to a single mode, in Sec. III the incident and scattered beams are described by completely arbitrary distributions over the modes. The mathematical procedures used in the two formulations have sufficient formal similarity that it can be readily seen that the more general results obtained from the stationary-state formulation apply also to the time-dependent formulation.

The general idea behind the theorems of Secs. II and III is the following. We wish to evaluate an element S_{fi} of the S matrix for an initial state i and a final state f using a creation and annihilation operator representation for the photons. For example, let i be a state having n photons in mode k, and with the scatterer described by an over-all quantum number α,

$$|i\rangle=(1/n!)^{1/2}(a_k^\dagger)^n|\alpha\rangle, \tag{1.2}$$

and let f be a state with $n-1$ photons in mode k, one photon in mode l, and with the scatterer in a state β,

$$|f\rangle=[1/(n-1)!]^{1/2}a_l^\dagger(a_k^\dagger)^{n-1}|\beta\rangle, \tag{1.3}$$

where the Dirac kets $|\alpha\rangle$ and $|\beta\rangle$ represent the photon vacuum in addition to the states α and β of the scatterer, and $a_k^\dagger$ and a_k are creation and annihilation operators, respectively. The S-matrix element is then written as

$$S_{fi}=\langle f|S|i\rangle=[1/n!(n-1)!]^{1/2}\times\langle\beta|a_k^{n-1}a_lS(a_k^\dagger)^n|\alpha\rangle. \tag{1.4}$$

Now, we wish to use the known commutation relations between the creation and annihilation operators and the operators comprising S to bring the creation operators through to the left and the annihilation operators through to the right. The result will be a matrix element of a new operator between the photon vacuum-state vectors $|\alpha\rangle$ and $|\beta\rangle$. The new operator will display explicitly the external photon vertices corresponding to emission and absorption of the "real" photons represented by $(a_k^\dagger)^n$ and $a_l^\dagger(a_k^\dagger)^{n-1}$.

II. TIME-DEPENDENT FORMULATION

Summary of Facts Relevant to Covariant S Matrix

The S matrix is given by

$$S=\sum_n\left(\frac{-i}{\hbar c}\right)^n\frac{1}{n!}\int_{-\infty}^{\infty}dx_1\cdots dx_n\times P[H(x_1)\cdots H(x_n)], \tag{2.1}$$

where P is the Dyson time-ordering operator, x is a space-time four vector, and dx is a space-time four differential. The interaction Hamiltonian density is

$$H(x)=j_\mu(x)A_\mu(x), \tag{2.2}$$

where $j_\mu(x)$ is the fermion four-current density operator, and $A_\mu(x)$ is the vector potential, whose plane-wave expansion for periodic quantization in a volume V is

$$A_\mu(x)=(\hbar c/V)^{1/2}\sum_{\mathbf{k}}(2\omega_k)^{-1/2}\times[a_\mu(\mathbf{k})e^{-ikx}+a_\mu^\dagger(\mathbf{k})e^{ikx}], \tag{2.3}$$

where

$$a_\mu(\mathbf{k})=\sum_\lambda \epsilon_\mu(\lambda,\mathbf{k})a(\lambda,\mathbf{k});$$

the $\epsilon_\mu(\lambda,\mathbf{k})$ are unit polarization vectors, and $a(\lambda,\mathbf{k})$ is an annihilation operator for a photon of polarization λ and wave vector $\mathbf{k}$. In (2.3) kx is a four-vector dot product. The creation and annihilation operators obey the commutation relation

$$[a_\mu(\mathbf{k}),a_{\mu'}{}^\dagger(\mathbf{k}')]=\delta_{\mu\mu'}\delta_{kk'}. \tag{2.4}$$

We shall need the commutator

$$J_\mu^\dagger(\mathbf{k},x)\equiv[a_\mu(\mathbf{k}),H(x)]=j_\mu(x)(\hbar c/V)^{1/2}(2\omega_k)^{-1/2}e^{ikx}. \tag{2.5}$$

For brevity we shall absorb the quantities μ and $\mathbf{k}$ into the single label k and write, instead of (2.5),

$$J_k^\dagger(x)\equiv[a_k,H(x)]. \tag{2.6}$$

Commutation Theorems

Now consider the time-independent operator A, and the set of time-dependent operators $B_\alpha(x)$ with the property

$$[A,B_\alpha(x)]=C_\alpha(x), \tag{2.7}$$

where C_α commutes with A. We wish to commute A through a product of B_α's. It follows from (2.7) that

$$AB_1\cdots B_N=B_1\cdots B_NA +\sum_r B_1\cdots B_{r-1}C_rB_{r+1}\cdots B_N,\quad r=1\to N, \tag{2.8}$$

as can easily be proved by induction. If the operators on the left-hand side of (2.8) are time ordered, the same is true of the right-hand side, including the placement of $C_r(x)$. Consequently, we can write

$$AP[B_1(x_1)\cdots B_N(x_N)]=P[B_1(x_1)\cdots B_N(x_N)]A +\sum_r P[B_1(x_1)\cdots B_{r-1}(x_{r-1})C_r(x_r) \times B_{r+1}(x_{r+1})\cdots B_N(x_N)],\quad r=1\to N. \tag{2.9}$$

Later it will be convenient to be able to write the time-independent operator A inside the time-ordering brackets, so we now extend the definition of P as follows. If A is any time-independent operator and D is *any* operator, then

$$\begin{aligned}P[AD]&\equiv AP[D],\\ P[DA]&\equiv P[D]A.\end{aligned} \tag{2.10}$$

If E is any operator which commutes with A, we further define

$$\begin{aligned}P[EAD]&\equiv P[AED]=AP[ED],\\ P[DAE]&\equiv P[DEA]=P[DE]A.\end{aligned} \tag{2.11}$$

Now, if the $F_\alpha(x_\alpha)$ below are operators which commute with a_k, it follows from (2.9) and the definition (2.6) of $J_k^\dagger$ that

$$\begin{aligned}a_k\int dx_1\cdots dx_{n+N}&P[H(x_1)\cdots H(x_n) \times F_1(x_{n+1})\cdots F_N(x_{n+N})]\\ =\int dx_1\cdots dx_{n+N}&P[H(x_1)\cdots H(x_n) \times F_1(x_{n+1})\cdots F_N(x_{n+N})a_k]\\ +n\int dx_1\cdots dx_{n+N}&P[H(x_1)\cdots H(x_{n-1}) \times F_1(x_n)\cdots F_N(x_{n-1+N})J_k^\dagger(x_{n+N})].\end{aligned} \tag{2.12}$$

Next let us make the definition

$$D_\alpha=\int dxF_\alpha(x). \tag{2.13}$$

According to the definitions (2.1) of S and (2.13) we have,

$$P[SD_1\cdots D_N]=\sum_n\left(\frac{-i}{\hbar c}\right)^n\frac{1}{n!}\int_{-\infty}^{\infty}dx_1\cdots dx_{n+N} \times P[H(x_1)\cdots H(x_n)F_1(x_{n+1})\cdots F_N(x_{n+N})]. \tag{2.14}$$

Equation (2.12) can now be used to yield

$$a_kP[SD_1\cdots D_N]=P[SD_1\cdots D_Na_k] +P[SD_1\cdots D_NK_k], \tag{2.15}$$

where we have used the definition

$$K_k\equiv\frac{-i}{\hbar c}\int dxJ_\mu^\dagger(x). \tag{2.16}$$

Successive applications of (2.15) lead directly to the general rule,

$$a_k{}^nP[SD_1\cdots D_N]=P[SD_1\cdots D_N(a_k+K_k)^n]. \tag{2.17}$$

The rule for bringing creation operators through from the right to the left, which can be found in a similar manner, is

$$P[SD_1'\cdots D_N'](a_k^\dagger)^n =P[(a_k^\dagger+\bar{K}_k)^nSD_1'\cdots D_N'], \tag{2.18}$$

where

$$\bar{K}_k\equiv\frac{-i}{\hbar c}\int dxJ_k(x), \tag{2.19}$$

$$D_\alpha'\equiv\int dxF_\alpha'(x), \tag{2.20}$$

and F_α' is an arbitrary operator which commutes with $a_k^\dagger$.

Example: Single-Photon Scattering

To illustrate how (2.17) and (2.18) are used let us apply them to the S matrix for scattering one photon into a mode labeled l, from a beam of n photons all in mode k with the fermion scatterer changing from state α to state β. The initial and final state vectors, $|i\rangle$ and $|f\rangle$, are given by (1.2) and (1.3), and the S-matrix element S_{fi} is given by (1.4). The operators a_l and $a_k{}^{n-1}$ can be brought through to the right with (2.17), the result can be simplified by means of the identity

$$a_k{}^m(a_k^\dagger)^n|\alpha\rangle=[n!/(n-m)!](a_k^\dagger)^{n-m}|\alpha\rangle, \tag{2.21}$$

and all remaining creation operators can be brought

through to the left using (2.18) to yield

$$S_{fi}=\sum_r [n!(n-1)!]^{1/2}[r!(r+1)!(n-1-r)!]^{-1} \times\langle\beta|P[S(\bar{K}_k)^{r+1}K_k{}^rK_l]|\alpha\rangle. \quad (2.22)$$

Now let us define

$$K_k=[(2\pi)^{3/2}/V^{1/2}]L_k, \quad \bar{K}_k=[(2\pi)^{3/2}/V^{1/2}]\bar{L}_k, \quad (2.23)$$

so that in the expanded notation of (2.5),

$$L_\mu(\mathbf{k})=\frac{-i}{\hbar c}\int dx j_\mu(x)\left[(2\pi)^{-\frac{3}{2}}\left(\frac{\hbar c}{2\omega_k}\right)^{1/2}e^{ikx}\right],$$
$$\bar{L}_\mu(\mathbf{k})=\frac{-i}{\hbar c}\int dx j_\mu(x)\left[(2\pi)^{-\frac{3}{2}}\left(\frac{\hbar c}{2\omega_k}\right)^{1/2}e^{-ikx}\right], \quad (2.24)$$

where

$$j_\mu(x)=\bar{\psi}(x)\gamma_\mu\psi(x). \quad (2.25)$$

In terms of L_k, etc., S_{fi} takes the form

$$S_{fi}=[(2\pi)^3n/V]^{1/2} \times\sum_r [(n-1)!/r!(r+1)!(n-1-r)!] \times[(2\pi)^3/V]^r\langle\beta|P[S(\bar{L}_k)^{r+1}L_k{}^rL_l]|\alpha\rangle, \quad (2.26)$$

where the dependences on the quantization volume V are now explicit. In the reduction of S_{fi} to normal form using Wick's theorem[3] the L_k, etc., provide the appropriate quantities for vertices connecting to external photon lines, and for fermion lines connecting to these vertices. In the normal product expansion the fermion operators in $j_\mu(x)$ are treated in precisely the usual manner, i.e., they are contracted with other fermion operators to form fermion propagators, or they remain uncontracted to provide wave functions corresponding to external fermion lines. The factors in square brackets in (2.24) then appear as vertex functions at the vertices defined by $\bar{\psi}(x)$ and $\psi(x)$. These factors are just the photon wave functions which are to be used in the conventional Feynman formalism at external photon vertices. L_k corresponds to a photon emitted into mode k and $\bar{L}_k$ to a photon absorbed from mode k. The additional factor $(-i/\hbar c)$ is needed to be consistent with the rule that in the perturbation expansion there must be one such factor for each vertex of the diagram in question.

When $S_r\equiv\langle\beta|P[S(\bar{L}_k)^{r+1}L_k{}^rL_l]|\alpha\rangle$ is expanded into a sum of normal products evaluated between the photon vacuum-state vectors $|\alpha\rangle$ and $|\beta\rangle$, the expansion of S provides every possible diagram in which the fermion goes from α to β, but the presence of the other operators requires that in each such diagram there must be $r+1$ vertices for absorption of photons from mode k, r vertices for emission of photons into mode k, and one vertex for emission of a photon into mode l. Since these vertices correspond to photon wave functions, rather than photon propagators, the corresponding photon lines must be external; that is, they cannot have their other ends attached to second vertices. Thus, it is convenient to think of these $2r+2$ photons as real, rather than virtual. Since the system is now to be thought of as starting in a photon vacuum α and ending in a photon vacuum β, the explicit appearance of the L_k, etc., constitutes the only way in which the real photons manifest their presence. There is no need to consider the effect of the real photons on the statistics of the virtual photons, since there are no operators in the matrix element to indicate the presence of the real photons. All such effects are explicitly accounted for by the factor

$$[(2\pi)^3n/V]^{1/2}[(n-1)!/r!(r+1)!(n-1-r)!] \times[(2\pi)^3/V]^r.$$

Since the system starts and ends in photon vacuum states in (2.26), all photon lines provided by the expansion of S must be virtual, i.e., both ends must be at vertices, and they must be summed over all modes.

In summary, all interactions with the real photons are indicated explicitly by the L and $\bar{L}$ operators, and S provides all possible radiative corrections. The latter are to be made as though there were no real photons present.

The transformation from K_k to L_k not only displays the volume dependences explicitly, but also puts the photon wave function in the form appropriate to normalization in a continuum. When the transition to the continuum is made using (1.1), the photon propagators coming from the commutators of the vector potential operators (2.3) have a factor V^{-1} and a summation over modes (the double summation immediately reduces to a single summation because of the Kronecker deltas coming from the commutators of the creation and annihilation operators) whose transformation to an integral via (1.1) introduces a V which cancels the V^{-1}. Consequently, there will be no volume dependences internal to the matrix element.

For any diagram of S_r having a given ordering of the external photon emission vertices (labeled by the r sets of space-time variables) there will be another topologically identical diagram for every possible permutation of the r vertices. This is because the space-time variables which do the labeling are merely dummy variables, and are integrated over. Consequently, S_r contains $r!(r+1)!$ topologically identical sets of diagrams, corresponding to the permutations of the identical external emission and identical external absorption vertices. If we now define the subscript 0 on a ket, $\langle\cdots\rangle_0$, to mean "enumerate only diagrams which are topologically distinct with respect to the external photon vertices," then we may write (2.26) as

$$S_{fi}=[(2\pi)^3n/V]^{1/2}\sum_r [(n-1)!/(n-1-r)!] \times[(2\pi)^3/V]^r\langle\beta|P[S(\bar{L}_k)^{r+1}L_k{}^rL_l]|\alpha\rangle_0. \quad (2.27)$$

We may think of each diagram in S_r as containing

[3] G. C. Wick, Phys. Rev. 80, 268 (1950).

the scattering of one photon from mode k to mode l, and r photons "forward scattering" from mode k back to mode k. We cannot think of the forward scatterings as conventional radiative corrections in which the mode in question happens to coincide with that occupied by the incident beam (as discussed in I), since they cannot occur an arbitrary number of times. The factor $(n-1)!/(n-1-r)!$ limits the number of external absorption vertices to n, the number of real photons.

The factor $(n/V)^{1/2}$ will be canceled when we take the absolute square of S_{fi} and divide by the incident flux $(n/V)c$ in calculating the cross section. Each term in S_r will then be multiplied by r factors

$$[(2\pi)^3(n-1)/V][(2\pi)^3(n-2)/V]\cdots[(2\pi)^3(n-r)/V].$$

If the total number n of photons remains finite as the quantization volume V approaches infinity, then all contributions except that of S_0 (zero forward scatterings) will vanish. However, if n also approaches infinity in such a way that the density $\rho=n/V$ of photons in the incident beam remains finite, then in general the factor in front of S_r will also remain finite. If, in the expression above, we take the limit of infinite n and V for fixed r the result is $[(2\pi)^3\rho]^r$. This suggests the general rule that a factor $[(2\pi)^3\rho]^{1/2}$ is to be associated with each vertex for absorbing a photon from or emitting a photon into a mode occupied by photons of number density ρ.

The above limiting process can be made more precise. Let us rewrite the expression (2.27) for the S-matrix element with an incident beam containing n photons as

$$S_{fi}(n)=\left[\frac{(2\pi)^3 n}{V}\right]^{1/2}\sum_{r=0}^{n-1}\frac{(n-1)!}{(n-1-r)!}\left[\frac{(2\pi)^3}{V}\right]^r A_r, \tag{2.28}$$

$$A_r=\langle\beta|P[S(\bar{L}_k)^{r+1}L_k{}^rL_l]|\alpha\rangle_0.$$

Now we must assume that, as n and V approach infinity such that $n/V=\rho$ remains fixed, the sequence $S_{fi}(n)$ approaches a limit L,

$$\lim_{n\to\infty}[S_{fi}(n)-L]=0. \tag{2.29}$$

No attempt will be made here to determine the conditions under which (2.29) holds. We merely point out that assuming (2.29) leads to the formal limit

$$\lim_{n\to\infty}S_{fi}(n)=[(2\pi)^3\rho]^{1/2}\sum_{r=0}^{\infty}[(2\pi)^3\rho]^r A_r. \tag{2.30}$$

In the next section we shall treat the circumstance in which the photons of the incident beam are distributed over a continuous range of modes. The need for the assumption (2.29) will be eliminated, because it will no longer be necessary to take the limit of an infinite number of photons in one mode. We shall find that (2.30) holds as the limiting case in which the range of modes occupied by the incident photons is so small that the densities of states and the matrix elements constituting the A_r in (2.30) are essentially constant over this range.

III. STATIONARY STATE FORMULATION DEFINITIONS

In order to evaluate S-matrix elements, we wish to be able to bring annihilation operators through from the left and creation operators through from the right of the T matrix, defined by

$$T(E)=H_1\sum_n[g(E)H_1]^n, \tag{3.1}$$

$$g(E)=(E-H_0+i\epsilon)^{-1}, \tag{3.2}$$

and related to the S matrix by[4]

$$S_{fi}=\delta_{fi}-2\pi i\delta(E_f-E_i)T_{fi}(E_i). \tag{3.3}$$

In the equations above, H_0 is the unperturbed or free-field Hamiltonian, H_1 is the interaction Hamiltonian, and E is to be thought of as an eigenenergy of the total Hamiltonian $H=H_0+H_1$.

To achieve this objective we define a permutation operator P for the product of Q's appearing inside the curly brackets $\{\cdots\}$ following it. The Q's are field operators of a particular type defined below. They are, in fact, a special case of the following general definition.

Let F be any field operator, i.e., an operator which is a functional of the boson creation and annihilation operators ($C-A$'s). We wish to provide a notation for higher order commutators of F with the latter. We define

$$F\equiv F(0|0), \tag{3.4a}$$

$$[a_{k_1},F(m_1k_1,m_2k_2,\cdots|n_1k_1,n_2k_2,\cdots)]\equiv F([m_1+1]k_1,m_2k_2,\cdots|n_1k_1,n_2k_2,\cdots), \tag{3.4b}$$

$$[F(m_1k_1,m_2k_2,\cdots|n_1k_1,n_2k_2,\cdots),a_{k_1}{}^\dagger]\equiv F(m_1k_1,m_2k_2,\cdots|[n_1+1]k_1,n_2k_2,\cdots). \tag{3.4c}$$

The Q operators are the special case for which

$$Q(0|0)=H_1. \tag{3.5}$$

Each Q is to be thought of as a vertex function for the emission of the number of bosons in each mode indicated to the left of the vertical line, and for the absorption of the number of bosons in each mode indicated to the right. The reason is that each commutation of H_1 removes a creation of annihilation operator, but leaves the appropriate c-number factor.

In addition to the Q's, $C-A$'s may also appear inside the P brackets, but the rules pertaining to them differ from those for the Q's. P has the following specific properties.

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[4] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

Property 1. P takes the sum of all possible permutations of the product of Q's inside the brackets.

Property 2. An operator $f(E,[\beta_k+\gamma_k])$ appears between any two adjacent Q's. f is a function having a set of arguments $[\beta_k+\gamma_k]$, one for each mode k of the boson field, plus an additional argument denoted above by E.

β_k is the number of emitted bosons minus the number of absorbed bosons in mode k counted from the vertex functions Q to the *right* of f, plus the total number of a_k's within the brackets.

E and the γ_k are arbitrary quantities, whose values are independent of the position of f. Where it is necessary they will be specified explicitly in round brackets after the permutation operator, as $P(E,[\gamma_k])$. Otherwise, they will be suppressed for notational brevity.

The operator f has two other specific properties:

$$a_k f(\cdots,\beta_k+\gamma_k,\cdots)=f(\cdots,\beta_k+\gamma_k+1,\cdots)a_k, \quad (3.6)$$

and

$$f^\dagger(E,[\beta_k+\gamma_k])=f(E^*,[\beta_k+\gamma_k]). \quad (3.7)$$

An example of particular interest is that for which f is taken to be

$$f(E+i\epsilon,[\beta_k+\gamma_k])=g[E-\textstyle\sum_k(\beta_k+\gamma_k)\omega_k], \quad (3.8)$$

where ω_k is the unperturbed energy of a boson in mode k. In this case, we shall see that the tallying property of β_k corrects the energy denominators for emitted or absorbed bosons indicated by the vertex functions Q. The fact that (3.6) is satisfied for this choice of the operator f can be seen without much difficulty by using the fact that $g(E)$ is diagonal with respect to the eigenstates of H_0.

A second possible choice of f is of importance in evaluating elements of the T matrix from which certain categories of Feynman diagrams have been excluded. For example, $\langle y|T|x\rangle$ is the sum of contributions from all possible diagrams for going from unperturbed state x to unperturbed state y, including diagrams with self-energy end modifications. The latter can be eliminated formally from the x end of the scatterer line by rewriting T as

$$T(E)=H_1\textstyle\sum_n[g(E)\Lambda H_1]^n, \quad (3.9)$$

where Λ is a projection operator off the state x,

$$\Lambda=1-|x\rangle\langle x|, \quad (3.10)$$

and prohibits the appearance of x as an intermediate state of the system in any diagram. The T matrix in the form (3.9) can be handled by choosing f to be

$$f(E+i\epsilon,[\beta_k+\gamma_k])=g[E-\textstyle\sum_k(\beta_k+\gamma_k)\omega_k]\times\Lambda([\beta_k+\gamma_k]), \quad (3.11)$$

where $\Lambda([\beta_k+\gamma_k])$ is a projection operator off that state whose occupation number for each mode k is $\beta_k+\gamma_k$ *smaller* than that of state x. If any such occupation number is negative, then $\Lambda([\beta_k+\gamma_k])$ is defined to be unity. It can be shown from the commutation relations of the $C-A$'s, and the $C-A$ representation of a normalized ket $|n_k\rangle$ for a state having occupation number n_k, that

$$\begin{aligned} a_k|n_k\rangle\langle n_k| &= |n_k-1\rangle\langle n_k-1|a_k, & n_k-1\geq 0 \\ &= \text{zero}, & n_k-1<0. \end{aligned} \quad (3.12)$$

Consequently, the required property (3.6) is possessed by the operator f defined in (3.11).

$C-A$'s appear inside the P brackets only for notational convenience. They are not permuted with the Q's. The rules relating to them are as follows.

Property 3. If X and Y are any products of Q's and $C-A$'s, then by definition

$$P\{Xa_kY\}\equiv P\{XYa_k\}, \quad (3.12a)$$

$$P(E,[\gamma_k])\{XYa_k\}\equiv P(E,[\gamma_k+1])\{XY\}a_k, \quad (3.12b)$$

and

$$P\{Xa_k^\dagger Y\}\equiv P\{a_k^\dagger XY\}, \quad (3.12c)$$

$$P\{a_k^\dagger XY\}\equiv a_k^\dagger P\{XY\}. \quad (3.12d)$$

In other words, regardless of where they may be written (for convenience) inside the brackets, all annihilation operators are to be thought of as taken out of the brackets to the right, and all creation operators to the left. The change in argument of P in (3.12b) is needed for consistency with the tallying rules of Property 2.

Finally, we include the additive property.

Property 4. If X and Y are any two products or sums of products of Q's and $C-A$'s, then

$$P\{X\}+P\{Y\}\equiv P\{X+Y\}. \quad (3.13)$$

Later we shall need the adjoint of a P product. If X below is any product of Q's and $C-A$'s this can be shown, as a consequence of the properties listed above, to be

$$[P(E,[\gamma_k])\{X\}]^\dagger=P(E^*,[\gamma_k+\lambda_k])\{X^\dagger\}, \quad (3.14)$$

where λ_k is the total number of emitted bosons minus the total number of absorbed bosons in mode k counted from the vertex functions Q, plus the total number of a_k's minus the total number of $a_k^\dagger$'s within the brackets (all before the adjoint is taken).

Note that, according to the definition (3.4), when the adjoint of a Q operator is taken absorptions become emissions and vice versa.

Commutation Theorems

If any two operators A and B_α have the commutator C_α,

$$[A,B_\alpha]=C_\alpha, \quad (3.15)$$

and if A and $f(\mu_\alpha)$ satisfy the relation

$$Af(\mu_\alpha)=f(\mu_\alpha+1)A, \quad (3.16)$$

then the following relation, which is the analog of (2.8), holds,

$$\begin{aligned}&AB_1f(\mu_1)B_2\cdots f(\mu_{N-1})B_N\\&=B_1f(\mu_1+1)B_2\cdots f(\mu_{N-1}+1)B_NA\\&\quad+\textstyle\sum_r B_1f(\mu_1+1)\cdots f(\mu_{r-1}+1)C_rf(\mu_r)\\&\qquad\cdots f(\mu_{N-1})B_N,\quad r=1\to N.\end{aligned}\tag{3.17}$$

Equation (3.16) is satisfied for $A=a_k$ and $\mu_\alpha=\beta_k+\gamma_k$, according to (3.6).

Now let D be any product of Q's, and consider the quantity

$$(1/n!)a_kP(E,[0])\{H_1{}^nD\},$$

which is a_k operating from the left on the sum of permutations of the factors comprising $H_1{}^nD$ with the appropriate f's; the $n!$ in the denominator cancels a similar factor resulting from the fact that for every arrangement of the operators $n!$ permutations of the H_1's are identical. If we apply (3.17) to each term in the sum, with $A=a_k$, n of the B_α's taken to be H_1's, and the rest of the B_α's taken to be the factors comprising D, we get

$$\begin{aligned}(1/n!)a_kP\{H_1{}^nD\}&=(1/n!)P\{H_1Da_k\}\\&\quad+[1/(n-1)!]P\{H_1{}^{n-1}DQ(k|0)\}\\&\quad+(1/n!)P\{H_1{}^nD(k|0)\}.\end{aligned}\tag{3.18}$$

Now, in terms of the definition

$$T'=\sum_{n=1}\frac{1}{n!}H_1{}^n,\tag{3.19}$$

we use (3.18) to find

$$\begin{aligned}a_kP\{(1+T')D\}&=P\{(1+T')[D(a_k+Q(k|0))\\&\quad+D(k|0)]\}.\end{aligned}\tag{3.20}$$

The significance of T' is that it is the appropriate operator for expressing the T matrix inside the P bracket,

$$T(E)=P\{T'\},\tag{3.21}$$

for either form (3.1) or form (3.9).

By repeated application of (3.20) we find that its generalization takes the form

$$\begin{aligned}&a_k{}^mP\{(1+T')D_0\}\\&\quad=P\{(1+T')\textstyle\sum_p A(m,p_1,p_2,\cdots)\\&\qquad\times D_n(a_k+Q_1)^{p_1}Q_2{}^{p_2}Q_3{}^{p_3}\cdots\},\end{aligned}\tag{3.22a}$$

where

$$n=m-(p_1+2p_2+2p_3+\cdots),\tag{3.22b}$$

$\sum_p$ means sum over p_1, p_2, etc., and where we have written Q_n for $Q(nk|0)$ and D_n for $D(nk|0)$. Applying (3.20) to (3.22) leads to the difference equation,

$$\begin{aligned}A(m,p_1,p_2,\cdots)&=A(m-1,\,p_1,\,p_2,\,\cdots)\\&\quad+A(m-1,\,p_1-1,\,p_2,\,\cdots)+\textstyle\sum_j(p_j+1)\\&\qquad\times A(m-1,\,\cdots p_j+1,\,p_{j+1}-1,\,\cdots),\end{aligned}\tag{3.23}$$

whose solution is

$$\begin{aligned}A(m,p_1,p_2,\cdots)&=m![(m-\textstyle\sum_j jp_j)\\&\quad\times!p_1!p_2!\cdots(1!)^{p_1}(2!)^{p_2}\cdots]^{-1},\end{aligned}\tag{3.24}$$

as can be checked by substitution into (3.23). Equation (3.24) satisfies the boundary condition

$$A(0,0,0,\cdots)=1,\tag{3.25}$$

and its uniqueness follows from the fact that the solution for a given m is obtained uniquely from that for $m-1$, so that $A(0,\cdots)$ determines $A(1,\cdots)$, etc. We may now write the rule for bringing annihilation operators through from the left as

$$\begin{aligned}&a_k{}^mP\{(1+T')D\}\\&\quad=P\{(1+T')\textstyle\sum_{pq}m![D(Mk|0)/M!]\\&\qquad\times\prod_j(1/p_j!)[Q(jk|0)/j!]^{p_j}\\&\qquad\times[1/(q-p_1)!]a_k{}^{q-p_1}\},\\&\quad M=m-(q-p_1)-\textstyle\sum_j jp_j,\quad j=1\to\infty.\end{aligned}\tag{3.26}$$

The rule for bringing creation operators through from the right is obtained by taking the adjoint of (3.26) with the aid of (3.14). We find

$$\begin{aligned}&P(E,[0])\{(1+T')D\}(a_k{}^\dagger)^m\\&\quad=P(E,\,\gamma_k=m,\,\gamma_{k'}=0,\,k'\neq k)\{(1+T')\\&\qquad\times\textstyle\sum_{pq}m![1/(q-p_1)!](a_k{}^\dagger)^{q-p_1}[D(0|Mk)/M!]\\&\qquad\times\prod_j(1/p_j!)[Q(0|jk)/j!]^{p_j}\},\\&\qquad j=1\to\infty,\end{aligned}\tag{3.27}$$

where M is defined as in (3.26).

Additional Theorem

We need one more theorem, as we shall find it necessary to be able to evaluate the P product of an arbitrary order commutator of $a_k{}^\dagger$ with a general product, where the nth order commutator with $a_k{}^\dagger$ is defined by

$$[F,a_k{}^\dagger]_n=[[F,a_k{}^\dagger]_{n-1},a_k{}^\dagger],\tag{3.28a}$$

$$[F,a_k{}^\dagger]_0=F.\tag{3.28b}$$

The desired theorem, which is a consequence of (3.17) and considerable commutator algebra, is

$$\begin{aligned}P\{[\textstyle\prod_r B_r{}^{p_r},a_k{}^\dagger]_n\}&=n!P\{\textstyle\sum_s\prod_r p_r!\prod_t(1/s_{rt}!)\\&\quad\times[B_r(0|tk)/t!]^{s_{rt}}\},\\&\qquad t=0\to\infty,\\n&=\textstyle\sum_{rt}ts_{rt},\\p_r&=\textstyle\sum_t s_{rt}.\end{aligned}\tag{3.29}$$

In (3.29) the symbol $\sum_s$ means sum over all s_{rt}.

We shall also need a corollary to (3.29) which comes directly from taking its adjoint,

$$\begin{aligned}P\{{}_n[a_k,\textstyle\prod_r B_r{}^{p_r}]\}&=n!P\{\textstyle\sum_s\prod_r p_r!\prod_t(1/s_{rt}!)\\&\quad\times[B_r(tk|0)/t!]^{s_{rt}}\},\\&\qquad t=0\to\infty\\n&=\textstyle\sum_{rt}ts_{rt},\\p_r&=\textstyle\sum_t s_{rt}.\end{aligned}\tag{3.30}$$

The nth order commutator with a_k is defined in analogy to (3.28) by

$$ {}_n[a_k,F]=[a_k,\ {}_{n-1}[a_k,F]], \tag{3.31a}$$

$$ {}_0[a_k,F]=F. \tag{3.31b}$$

Evaluation of the T Matrix

Let us now evaluate the T-matrix element for an initial state i of the system represented by the state vector

$$ |i\rangle=\prod_\mu (n_\mu!)^{-1/2}(a_{k_\mu}{}^\dagger)^{n_\mu}|x\rangle, \tag{3.32a}$$

and a final state f represented by

$$ |f\rangle=\prod_\nu (m_\nu!)^{-1/2}(a_{k_\nu}{}^\dagger)^{m_\nu}|y\rangle, \tag{3.32b}$$

where $|x\rangle$ and $|y\rangle$ are state vectors for the boson vacuum as well as for the internal states x and y of the scatterer. The subscripts μ and ν run over all modes of the boson field, so n_μ and m_ν represent completely general specifications of the initial and final-state boson distributions. At this stage, the quantization is considered to be periodic in a volume V, so μ and ν take only discrete values. By using (3.21) we can write the desired T-matrix element as

$$ T_{fi}+\delta_{fi}=\langle y|\prod_\nu (m_\nu!)^{-1/2}(a_{k_\nu})^{m_\nu}P(E_i[0]) \times\{1+T'\}\prod_\mu (n_\mu!)^{-1/2}(a_{k_\mu}{}^\dagger)^{n_\mu}|x\rangle, \tag{3.33}$$

where E_i is the total energy of the system in state i,

$$ E_i=E_0+\sum_\mu n_\mu\omega_{k_\mu}, \tag{3.34}$$

E_0 being the true initial energy of the isolated scatterer.

Now, by using (3.26) to bring the annihilation operators through from the left, (3.27) to bring the creation operators through from the right, and (3.29), (3.30) to evaluate resulting general order commutators, we ultimately arrive at the following expression for T_{fi}:

$$ \begin{aligned} T_{fi}+\delta_{fi}=&\sum_{\alpha s}\prod_\mu [m_\mu!n_\mu!/(m_\mu-\epsilon_\mu)!(n_\mu-\alpha_\mu)!]^{1/2} \\ &\times\langle y|P(E_i,[n_\mu])\{(1+T') \\ &\times\prod_{rt}{}'[1/s(r_1\cdots t_1\cdots)!] \\ &\times[Q(r_1k_1,\cdots|t_1k_1,\cdots)/ \\ &\qquad r_1!\cdots t_1!\cdots]^{s(r_1\cdots t_1\cdots)}\}|x\rangle, \end{aligned} \tag{3.35a}$$

$$ \alpha_\mu=\sum_{rt} t_\mu s(r_1\cdots t_1\cdots), \tag{3.35b}$$

$$ \epsilon_\mu=\sum_{rt} r_\mu s(r_1\cdots t_1\cdots), \tag{3.35c}$$

$$ n_\mu-m_\mu=\alpha_\mu-\epsilon_\mu. \tag{3.35d}$$

In (3.35) r, t, α, s under the summation and product symbols mean sum or take products over r_1, r_2, etc., t_1, t_2, etc., α_1, α_2, etc., and all $s(r_1\cdots t_1\cdots)$. The prime on the product symbol $\prod'$ means that r_1, r_2, etc., t_1, t_2, etc., are not to be taken simultaneously zero. α_μ is the total number of boson absorptions in mode μ counted from the Q's, and ϵ_μ is the total number of boson emissions in mode μ counted from the Q's, for given values of the quantities $s(r_1\cdots t_1\cdots)$.

Let us now define vertex functions which will be more convenient than the Q's in the limit of infinite quantization volume V,

$$ U(r_1k_1,\cdots|t_1k_1,\cdots)=[V/(2\pi)^3]^{(\Sigma r_\mu+\Sigma s_\mu)/2} \times Q(r_1k_1,\cdots|t_1k_1,\cdots)/r_1!\cdots t_1!\cdots. \tag{3.36}$$

The U's have no volume dependence, since each Q implicitly contains a factor $V^{-1/2}$ for each emitted or absorbed boson. This is a consequence of the fact that if H_1 is of n-fold linearity in the boson field operators, then its expansion in the boson creation and annihilation operators is an n-fold Fourier expansion in momentum or wave-vector space and must, therefore, have a normalization factor $V^{-n/2}$. For each emitted or absorbed boson, H_1 has been commuted with an annihilation or a creation operator, thereby reducing a summation over all modes to a single term, and leaving a factor $V^{-1/2}$ which no longer serves to normalize a Fourier expansion. This transformation from the Q's to the U's is the analog of the transformation (2.23) from the K's to the L's in Sec. II.

In Eq. (3.35) a factor like

$$ [Q(r_1k_1,\cdots|t_1k_1,\cdots)]^{s(r_1\cdots t_1\cdots)}$$

leads to $s(r_1\cdots t_1\cdots)!$ topologically identical diagrams because of the permutations that must be taken of the $s(r_1\cdots t_1\cdots)$ factors. Consequently, it is again convenient to use the notation $\langle\cdots\rangle_0$, which means that only topologically distinct diagrams are to be enumerated, and rewrite (3.35) as

$$ \begin{aligned} T_{fi}+\delta_{fi}=&\sum_{\alpha s}\prod_\mu([m_\mu!n_\mu!/(m_\mu-\epsilon_\mu)!(n_\mu-\alpha_\mu)!] \\ &\times[(2\pi)^3/V]^{\alpha_\mu+\epsilon_\mu})^{1/2}\langle y|P(E_i,[n_\mu])\{(1+T') \\ &\times\prod_{rt}{}'[U(r_1k_1,\cdots|t_1k_1,\cdots)]^{s(r_1\cdots t_1\cdots)}\}|x\rangle_0, \end{aligned} \tag{3.37}$$

with (3.35b, c, d) remaining appropriate.

Because of the explicitly indicated arguments of $P(E_i,[n_\mu])$, the f operators (3.11) appearing in (3.37) have the form

$$ f(E_i+i\epsilon,[\beta_\mu+n_\mu]) =g(E_0-\sum_\mu\beta_\mu\omega_\mu)\Lambda([\beta_\mu+n_\mu]). \tag{3.38}$$

The energy denominators are now referred to the energy E_0 of the *isolated* scatterer. Similarly, the projection operators are now referred to a state whose occupation numbers have been reduced from those of the original excluded state by n_μ, the number of bosons per mode of the incident beam. This compensates for the fact that formally in (3.37) the initial state of the system is x, the original initial state with the incident bosons removed. The U's do not have creation or annihilation operator properties with respect to the bosons for which they serve as vertex functions. Consequently, they change the state of the system only insofar as the *diagram* is concerned. The tallying properties of the β_μ in $\Lambda([\beta_\mu+n_\mu])$ clearly account for this in such a way that the original excluded state

continues to be prohibited from appearing in *diagrams*.

The U operators in (3.37) provide *all* the external boson lines for the Feynman diagrams. All such lines corresponding to a given U are to be drawn emanating from the same point. Virtual boson lines may emanate from the same point as external boson lines if the number of external lines is less than the order of linearity of H_1 in the boson field operators. The presence of the T' operator allows for all possible radiative corrections to the basic diagrams. As in the time-dependent formalism the radiative corrections are to be made as though the incident beam were absent. The presence of the latter is manifested in (3.37) only by the c-number portions of the U operators and by the explicit statistical factor in front.

Transition to the Continuum

Let $\mu=1$ to A label the modes from which bosons are absorbed (absorption modes), i.e., the modes for which $n_\mu-m_\mu>0$, and let $\mu=A+1$ to $A+B$ label the modes into which bosons are emitted (emission modes), i.e., the modes for which $n_\mu-m_\mu<0$. For all other modes we have $n_\mu=m_\mu$ so that according to (3.35d) we have also $\alpha_\mu=\epsilon_\mu$, i.e., for these modes every absorption is paired with an emission. We shall refer to these paired emissions and absorptions in the same mode as "forward scatterings." For the absorption modes every absorption in excess of $n_\mu-m_\mu$ can be thought of as paired with an emission, since according to (3.35d) the number of emissions is $\epsilon_\mu=\alpha_\mu-(n_\mu-m_\mu)$. Similarly, for the emission modes every emission in excess of $m_\mu-n_\mu$ can be thought of as paired with an absorption. We shall refer also to these absorption-emission pairs as "forward scatterings."

The infinite series of terms indicated in (3.37) can be arranged in order of increasing number of external absorption and emission lines. Each "basic" diagram, i.e., each diagram having only external boson lines, is of course accompanied by an infinite number of diagrams corresponding to all possible radiative corrections. Each order in this arrangement, then, is actually composed of an infinite class of diagrams characterized by a basic diagram. In the lowest order there are no forward scatterings. The lowest order contribution to $T_{fi}+\delta_{fi}$ is then given by

$$K\sum_s \langle y|P\{(1+T')\prod_{rt}{}'[U(r_{A+1}k_{A+1},\cdots r_{A+B}k_{A+B}|t_1k_1,\cdots t_Ak_A)]^{s(t_1\cdots r_{A+B})}\}|x\rangle_0, \quad (3.40)$$

where K is defined by

$$K=\prod_{\mu=1}^{A}\left[\frac{n_\mu!}{m_\mu!}\left[\frac{(2\pi)^3}{V}\right]^{n_\mu-m_\mu}\right]^{1/2} \times\prod_{\nu=A+1}^{A+B}\left[\frac{m_\nu!}{n_\nu!}\left[\frac{(2\pi)^3}{V}\right]^{m_\nu-n_\nu}\right]^{1/2}, \quad (3.41)$$

with the restriction that for any diagram the vertex functions U must be chosen in such a way that the total number of external absorption lines is equal to $n_\mu-m_\mu$ for each absorption mode, and the total number of external emission lines is equal to $m_\mu-n_\mu$ for each emission mode.

The statistical quantity K is common to all orders, since it is independent of the α_μ and ϵ_μ. If it is divided out of the general statistical factor in (3.37), the remaining factors are functionally different for the absorption and emission modes than they are for the other modes. However, no special treatment need be given to the absorption and emission modes, since, as we indicate below, in the continuum limit their contribution vanishes in higher orders. In this limit, which is achieved by taking $V\to\infty$, the number of bosons per mode remains bounded, but the density of modes in wave-vector space becomes infinite. That is, although the total number of modes and the total number N of bosons becomes infinite, the n_μ and m_μ remain bounded in such a way that the (finite) over-all density ρ of incident bosons is given by

$$\rho=N/V=(1/V)\sum_\mu n_\mu \underset{V\to\infty}{\longrightarrow}(2\pi)^{-3}\int d^3k n(k), \quad (3.42)$$

where (1.1) has been used, and the integration symbol is meant to imply also a summation over spin states or polarizations. (The latter convention will be assumed in all of the following equations.) Therefore, since the statistical factor provides a V in the denominator of each term, and the numerator, involving n_μ and m_μ remains finite, and since the total number of absorption and emission modes is finite, it follows that in the limit $V\to\infty$ their contributions vanish individually and cumulatively. Similarly, whenever two or more forward scatterings are constrained to the same mode there will always be two or more $(1/V)$'s in the appropriate statistical factor, but only one summation to contribute a V via the density of states as in (1.1), so the entire contribution will vanish in the limit of infinite volume. Consequently, any given mode μ can be involved in at most one forward scattering, so that we always have $\alpha_\mu=\epsilon_\mu=1$, and the statistical factor is always $(2\pi)^3n_\mu/V$. As a result, the expression for $T_{fi}+\delta_{fi}$ including contributions from diagrams of all orders, i.e., any number of forward scatterings can be written as

$$T_{fi}+\delta_{fi}=K\sum_l\int d^3q_1 n(\mathbf{q}_1)\cdots\int d^3q_l n(\mathbf{q}_l) \times\sum_s\langle y|P\{(1+T') \times\prod_{rt}{}'[U(R,\mathbf{q}_1,\cdots\mathbf{q}_l|T,\mathbf{q}_1,\cdots\mathbf{q}_l)]^{s(t_1\cdots r_{A+B})}\}|x\rangle_0, \quad l=0\text{ to }\infty, \quad (3.42)$$

where for brevity we have adopted the notation

$$U(R,\cdots|T,\cdots)=U(r_{A+1}k_{A+1},\cdots r_{A+B}k_{A+B},\cdots|t_1k_1,\cdots t_Ak_A,\cdots), \quad (3.43)$$

and where the restriction continues to apply that the total number of external lines in each absorption or emission mode is $n_\mu - m_\mu$ and $m_\mu - n_\mu$, respectively.

The transition rate R_{fi} involves a summation over final states,

$$R_{fi} = (2\pi/\hbar)\sum_f |T_{fi}|^2 \delta(E_f - E_i), \tag{3.44}$$

which in turn implies a summation for each of the modes $\mu = A+1$ to $A+B$ into which the final state bosons are emitted, as well as a summation for each of the modes $\mu = 1$ to A *from* which the initial state bosons are removed. Consequently, there are $A+B$ summations, each of which is converted into a volume integral in wave-vector space via (1.1). By considerations of the volume dependences similar to those above we find that if more than one boson is constrained to be absorbed from or emitted into any one of these modes $\mu = 1$ to $A+B$, then R_{fi} must vanish as V approaches infinity. The density of states, then, predominates over the statistics, and the occupation number of any mode can change by no more than unity. The transition rate becomes $(f \neq i)$

$$\begin{aligned} R_{fi} = (2\pi/\hbar) &\int_{\Omega_1} d^3k_1 n(\mathbf{k}_1) \cdots \int_{\Omega_A} d^3k_A n(\mathbf{k}_A) \\ &\times \int_{\Omega_{A+1}} d^3k_{A+1} [n(\mathbf{k}_{A+1}) + 1] \\ &\cdots \int_{\Omega_{A+B}} d^3k_{A+B} [n(\mathbf{k}_{A+B}) + 1] \\ &\times \left| \sum_{l=0}^{\infty} \int d^3q_1 n(\mathbf{q}_1) \cdots \int d^3q_l n(\mathbf{q}_l) \right. \\ &\left. \times \sum_{sy} \langle y | P\{(1+T') \textstyle\prod_{rt}{}' U^s\} | x \rangle_0 \right|^2 \\ &\times \delta(E_f - E_i), \end{aligned} \tag{3.45}$$

$$U^s = [U(r_{A+1}\mathbf{k}_{A+1}, \cdots r_{A+B}\mathbf{k}_{A+B}, \mathbf{q}_1, \cdots \mathbf{q}_l | t_1\mathbf{k}_1, \cdots t_A\mathbf{k}_A, \mathbf{q}_1, \cdots \mathbf{q}_l)]^{s(t_1 \cdots r_{A+B})},$$

where Ω_1 to Ω_{A+B} are the desired ranges of integration for the wave vectors of the emitted and absorbed bosons.

Equation (3.45) provides the rules, in their most general form, for associating statistical weighting functions with Feynman diagrams. For every external boson line which is counted as part of a "forward scattering" pair the T-matrix term is to have a factor $n^{1/2}$; for every external boson line representing an absorption from the incident beam, but not counted among the forward scatterings, there is to be a factor $n^{1/2}$; for every external boson line representing an emission not counted among the forward scatterings, i.e., for an emission into a final-state mode, there is to be a factor $(n+1)^{1/2}$.

The integrations over final-state ranges Ω_{A+1} to Ω_{A+B} can include the range occupied by the incident beam, in which case $n(\mathbf{k}_{A+1})$, etc., are not zero. Transitions for which this is the case do not constitute forward scattering in the strict sense of the term, which implies identity of the initial and final states. These correspond to events in which incident bosons scatter out of their original modes into *other* modes occupied by the beam.

In the appropriate circumstances (3.45) reduces to a form in which the weighting factors are the same as those obtained in Sec. II, where it is assumed that all of the incident bosons occupied the same mode. Let us assume now that the incident beam occupies a range of wave-vector space δ^3k about some $\mathbf{k}_0$, which is so narrow that (a) the matrix element $\langle y | \cdots | x \rangle_0$ is essentially constant over δ^3k, (b) the densities of final states, which are functionally dependent on the incident wave vector through the energy conserving delta function $\delta(E_f - E_i)$, are also insensitive to a variation of this wave vector over δ^3k, and, (c) the ranges Ω_{A+1} to Ω_{A+B} do not include δ^3k. The integration ranges Ω_1 to Ω_A can be taken to span that occupied by the incident beam, and the corresponding integrals can be performed using (3.42) to give

$$\begin{aligned} R_{fi} = (2\pi/\hbar)[(2\pi)^3\rho]^A &\int_{\Omega_{A+1}} d^3k_{A+1} \\ &\cdots \int_{\Omega_{A+B}} d^3k_{A+B} \Big| \sum_{l=0}^{\infty} [(2\pi)^3\rho]^l \\ &\times \sum_{sy} \langle y | P\{(1+T') \textstyle\prod_{rt}{}' U^s\} | x \rangle_0 |^2 \\ &\times \delta(E_f - E_i), \end{aligned} \tag{3.46}$$

$$U^s = [U(r_{A+1}\mathbf{k}_{A+1}, \cdots r_{A+B}\mathbf{k}_{A+B}, \mathbf{k}_0{}^{(1)}, \cdots \mathbf{k}_0{}^{(l)} | t_1\mathbf{k}_1, \cdots t_A\mathbf{k}_A, \mathbf{k}_0{}^{(1)}, \cdots \mathbf{k}_0{}^{(l)})]^{s(t_1 \cdots r_{A+B})},$$

where the superscripts on $\mathbf{k}_0{}^{(1)}$ to $\mathbf{k}_0{}^{(l)}$ merely serve to indicate the number of times $\mathbf{k}_0$ appears. In this expression each T-matrix term is weighted by a factor $[(2\pi)^3\rho]^{1/2}$ for every external line corresponding to a boson in δ^3k. It should be noted that the quantity

$$U(\cdots \mathbf{k}_0{}^{(1)}, \cdots \mathbf{k}_0{}^{(l)} | \cdots \mathbf{k}_0{}^{(1)}, \cdots \mathbf{k}_0{}^{(l)}) = \lim_{\mathbf{k}_1, \mathbf{k}_2, \cdots \to \mathbf{k}_0} U(\cdots \mathbf{k}_1, \cdots \mathbf{k}_l | \cdots \mathbf{k}_1, \cdots \mathbf{k}_l)$$

is not the same as $U(\cdots l\mathbf{k}_0 | \cdots l\mathbf{k}_0)$ for $l > 1$. This statement follows from the definition (3.36) and the fact that $Q(\cdots \mathbf{k}_0{}^{(1)}, \cdots \mathbf{k}_0{}^{(l)} | \cdots \mathbf{k}_0{}^{(1)}, \cdots \mathbf{k}_0{}^{(l)})$ is equal to $Q(\cdots l\mathbf{k}_0 | \cdots l\mathbf{k}_0)$.

Finally, it should be pointed out that the basic theorems (3.26), (3.27), (3.29) and (3.30) have formal

analogs in which the permutation operator is replaced by the Dyson time-ordering operator and $1+T'$ is replaced by S in the form (2.1) but without the restriction to an interaction Hamiltonian which is linear in the boson field operators. The derivations of the analog theorems are almost step-by-step identical to the derivations in Sec. III. As a consequence of this formal analogy, the general weighting functions indicated in Eq. (3.45) are also appropriate in the time-dependent formulation.

Summary

It has been demonstrated that in evaluating Feynman diagrams for elements of the S matrix in boson field theories the radiative corrections are to be calculated as though the incident beam were absent. The presence of the incident bosons is manifested by intensity-dependent weighting functions which are consequences of the Bose statistics, and which depend on the count of only external boson lines connected to the scatterer line. A prescription has been derived for the weighting functions, which takes the following form in the continuum limit. For every external boson line which is counted as part of a "forward scattering" pair, the T-matrix term is to have a factor of the square root $[n(\mathbf{k})]^{1/2}$ of the density in phase space of bosons at wave number $\mathbf{k}$; for every external boson line representing an absorption from the incident beam, but not counted among the forward scatterings, there is to be a factor $[n(\mathbf{k})]^{1/2}$; for every external boson line representing an emission not counted among the forward scatterings, i.e., for an emission into a final-state mode, there is to be a factor $[n(\mathbf{k})+1]^{1/2}$. The validity of the above statements has been demonstrated for all orders of the perturbation theory, and for both the time-dependent and the stationary-state Brillouin-Wigner perturbation developments with no restrictions being necessary on the nature of the interaction Hamiltonian. In addition it has been shown that in the Brillouin-Wigner formulation the exclusion of arbitrary intermediate states from the T-matrix expression does not alter any of the conclusions.

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